

Stochastic Analysis with Modelled Distributions

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Abstract

We introduce Sobolev-Slobodeckij type norms on spaces of modelled distributions in the framework of Martin Hairer’s regularity structures. We show directly that on these spaces reconstruction is still possible and we prove furthermore that those spaces are of martingale type 2 or UMD, which guarantees a rich stochastic integration theory for stochastic processes with values therein.

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1 Introduction

Modelled distributions are the spine of the theory of regularity structures [Hai14]: they constitute a way to describe, by means of functions taking values in a graded vector space which satisfy certain graded estimates, generalized functions of certain degrees of (ir-)regularity (which is hard-coded in the given graded vector space structure). Modelled distributions are mapped by the reconstruction operator to generalized functions on space time, which is the assertion of the celebrated reconstruction theorem. So far modelled distributions come with Hölder type norms, which is most natural from the point of view of the reconstruction theorem (see Theorem 3.10 in [Hai14]). However, with stochastic analysis in mind, Sobolev-Slobodeckij type norms are a more natural choice. It is the goal of this article to show that reconstruction still works by a direct proof, which, of course, mimics Martin Hairer’s original proof on the existence of the reconstruction operator.

The reconstruction operator \mathcal{R} maps modelled distributions to generalized functions in a linear and bounded way with additional continuous dependence on the underlying model. The reconstruction operator can be considered as an abstract integration operation, which depends on the particular regularity structure. It generalizes Young integration [You36] and controlled rough paths [Lyo98, Gub04], etc. The main result of this article can be seen as a

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Fubini theorem, which asserts for bounded modelled distribution valued predictable processes H and Brownian motion W that the order of “integration” can be interchanged

$$\left\langle \mathcal{R}((H \bullet W)), \psi \right\rangle = \left(\langle \mathcal{R}(H), \psi \rangle \bullet W \right)$$

for every test function ψ . This Fubini theorem has a deeper meaning if the space of modelled distributions \mathcal{D}_p^γ has, e.g., some martingale type 2 structure, such that a rich stochastic integration theory is at hand. There are many approaches to stochastic integration for Banach space valued processes, some of them involve properties of the Banach space like martingale type 2 or UMD. It depends on the purpose in mind, which property is actually needed, but for integrals with respect to Brownian motion martingale type 2 or UMD is favorable. We shall prove here that the space of modelled distributions \mathcal{D}_p^γ (for $p \geq 2$) are of martingale type 2 or UMD, respectively, which suffices to define a rich stochastic integration theory as needed for the treatment of stochastic differential equations with Brownian drivers like in the books of Da Prato-Peszat-Zabczyk [PZ07, DZ14].

A considerably more general investigation of Besov spaces of modelled distributions has been very recently and independently presented by [HL16]. From the embedding theorems therein one can partially deduce the reconstruction theorem of this paper, our proof might be of some independent interest due to its direct approach. Of course our results extend to those spaces, too, see Theorem 3.2.

In the sequel we shall shortly introduce some notation with respect to regularity structures, however, for the sake of readability we stay with standard Euclidean distances and do not introduce scaled distances. Of course all results generalize immediately to this situation. Subsection 2.1 provides the Sobolev-Slobodeckij type modelled distributions and the corresponding reconstruction operator. The Banach space properties of this new space of modelled distributions are established in Section 3.

Important notations: For two real functions a, b depending on variables x one writes $a \lesssim b$ if there exists a constant $C > 0$ such that $a(x) \leq C \cdot b(x)$, for all x , and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$ hold simultaneously. By $\lfloor a \rfloor$ for a number $a \in \mathbb{R}$ we mean $\lfloor a \rfloor := \sup\{b \in \mathbb{Z} : b \leq a\}$. The ball in \mathbb{R}^d , $d \in \mathbb{N}$, around $x \in \mathbb{R}^d$ with radius $R > 0$ is denoted by $\mathcal{B}_R(0)$.

The space of Hölder continuous functions $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ of order $r \geq 0$ is denoted by \mathcal{C}^r , that is, φ is bounded if $r = 0$, Hölder continuous for $0 < r \leq 1$ (which amounts precisely to Lipschitz continuous for $r = 1$, the derivative does not necessarily exist). For $r > 1$ not an integer the function is $\lfloor r \rfloor$ -times continuously differentiable and the derivatives of order $\lfloor r \rfloor$ are Hölder continuous of order $r - \lfloor r \rfloor$. For integers $r > 1$ the $(r-1)$ -th derivative exists and is Lipschitz continuous. If a function $\varphi \in \mathcal{C}^r$ has compact support, we say $\varphi \in \mathcal{C}_0^r$.

Additionally, we use $\varphi \in \mathcal{B}^r$ if $\varphi \in \mathcal{C}_0^r$ is such that $\|\varphi\|_{\mathcal{C}^r} \leq 1$ and $\text{supp } \varphi \subset \mathcal{B}_1(0)$. For a function $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ and given $\delta > 0$ and $x \in \mathbb{R}^d$, the scaling operator \mathcal{S}_x^δ is defined by $(\mathcal{S}_x^\delta \varphi)(y) := \varphi_x^\delta(y) := \delta^{-d} \varphi(\delta^{-1}(y - x))$. As usual $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^d)$ stands for the space of smooth functions $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ and \mathcal{C}_0^∞ is the subspace of all smooth functions with compact support. The dual space of

\mathcal{C}_0^∞ is the space of distributions denoted by $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^d)$. The zero is included in our notation of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and for a multi-index $k \in \mathbb{N}^d$, set $|k| := |(k_1, \dots, k_d)| := k_1 + \dots + k_d$ and $k! := k_1 \dots k_d$. The space $L^p := L^p(\mathbb{R}^d, \mathbb{R})$, $p \geq 1$, is the usual Lebesgue space, that is, the space of functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^d} |f(x)|^p dx < \infty$.

2 Modelled Distributions beyond Hölder type Norms

Recently, Martin Hairer introduced the theory of regularity structures (see [Hai14]). A gentle introduction to this novel theory can be found in [FH14] or [Hai15]. We recall here for the sake of completeness the fundamental objects in suitable generality for the present paper. For the convenience of the reader our notation and definitions are mainly borrowed from [Hai14, Hai15]. Let us start with the definition of a *regularity structure*, of its *models* and of *modelled distributions*.

Definition 2.1 (Definition 2.1 in [Hai14]). A triplet $\mathcal{T} = (A, T, G)$ is called *regularity structure* if it consists of the following objects:

- An *index set* $A \subset \mathbb{R}$, which is locally finite and bounded from below, with $0 \in A$.
- A *model space* $T = \bigoplus_{\alpha \in A} T_\alpha$, which is a graded vector space with each T_α a Banach space and $T_0 \approx \mathbb{R}$. Its unit vector is denoted by $\mathbf{1}$.
- A *structure group* G consisting of linear operators acting on T such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $a \in T_\alpha$ it holds

$$\Gamma a - a \in \bigoplus_{\beta \in A; \beta < \alpha} T_\beta.$$

Moreover, $\Gamma \mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

For any $\tau \in T$ and $\alpha \in A$ we denote by $\mathcal{Q}_\alpha \tau$ the projection of τ onto T_α and set $\|\tau\|_\alpha := \|\mathcal{Q}_\alpha \tau\|$.

The basic idea behind the *model space* T is to represent abstractly the information describing the “jet” or “local expansion” of a (generalized) function at any given point, i.e. we prescribe a certain structure of local expansions of (generalized) functions, which we have in mind. Each T_α then corresponds to the “monomials of degree α ” which are required to describe a (generalized) function locally “of order α ” and the role of the structure group G is to translate coefficients from a local expansion around a given point into coefficients for an expansion around another point, such that the (generalized) function does not change. To make this interpretation clearer, we present the abstract polynomials as very simple example of a regularity structure. A more detailed discussion of this example can be found in Section 2.2 in [Hai14] or Section 13.2.1 in [FH14]. Alternatively, the reader might keep in mind the theory of (controlled) rough paths [Lyo98, Gub04] as an example of a regularity structure, see Section 13.2.2 in [FH14].

Example 2.2. The polynomial regularity structure $T = \mathbb{R}[X_1, \dots, X_d]$ is given by the space of abstract polynomials in d variables. In this case the index set is the set of natural numbers, that is $A = \mathbb{N}$. The model space T is indeed a graded vector space since it can be written as

$$T = \bigoplus_{\alpha \in A} T_\alpha \quad \text{with} \quad T_\alpha := \text{span} \{X^k : |k| = \alpha\},$$

where $\text{span} \{X^k : |k| = \alpha\}$ is the space generated by all monomials of degree α and $X^k := X^{k_1} \dots X^{k_d}$. The canonical group action is $G \sim (\mathbb{R}^d, +)$ which acts on T via $\Gamma_h P(X) := P(X + h\mathbf{1})$ for every $h \in \mathbb{R}^d$ and $P(X) \in T$.

In order to associate to each “abstract” element in T a “concrete” (generalized) function or distribution on \mathbb{R}^d , Martin Hairer introduced the concept of *models*. For technical simplicity we consider here only the Euclidean scaling on \mathbb{R}^d , see for instance Section 2.3 in [Hai14] for all details.

Definition 2.3 (Definition 2.17 in [Hai14]). Given a regularity structure $\mathcal{T} = (A, T, G)$, a *model* (Γ, Π) on \mathbb{R}^d is given by:

- A linear map $\Gamma: \mathbb{R}^d \times \mathbb{R}^d \rightarrow G$ such that $\Gamma_{x,y} \Gamma_{y,z} = \Gamma_{x,z}$ for every $x, y, z \in \mathbb{R}^d$ and $\Gamma_{x,x} = 1$, where 1 is the identity operator.
- A collection of continuous linear maps $\Pi_x: T \rightarrow \mathcal{D}'$ such that $\Pi_y = \Pi_x \circ \Gamma_{x,y}$ for every $x, y \in \mathbb{R}^d$.

Furthermore, for every compact set $\mathcal{K} \subset \mathbb{R}^d$ and for every constant $\gamma > 0$, there exists a constant $C_{\gamma, \mathcal{K}} > 0$ such that the bounds

$$|\langle \Pi_x \tau, \varphi_x^\lambda \rangle| \leq C_{\gamma, \mathcal{K}} \lambda^\alpha \|\tau\|_\alpha \quad \text{and} \quad \|\Gamma_{x,y} \tau\|_\beta \leq C_{\gamma, \mathcal{K}} |x - y|^{\alpha - \beta} \|\tau\|_\alpha \quad (1)$$

hold uniform over $(x, y) \in \mathcal{K} \times \mathcal{K}$, $\lambda \in (0, 1]$, $\tau \in T_\alpha$ for $\alpha \leq \gamma$ and $\beta < \alpha$, and for all $\varphi \in \mathcal{B}^r$ with $r > |\inf A|$. Additionally, we denote by $\|\Pi\|_{\gamma, \mathcal{K}}$ the smallest constant for which the first inequality in (1) holds.

To illustrate the definition of a model, let us come back to Example 2.2.

Example 2.4. Given the polynomial regularity structure $\mathcal{T} = (A, T, G)$ from Example 2.2, a corresponding model (Γ, Π) can be defined by the concrete polynomials on \mathbb{R}^d . More precisely, the model (Γ, Π) is given by the action

$$\Gamma_{x,y} P(X) := P(X + (x - y)) \quad \text{and} \quad \Pi_x P(X) := P(\cdot - x),$$

for $X \in T$ and $x, y \in \mathbb{R}^d$.

The functions which can be described by the polynomial regularity structure and the model as introduced in Example 2.2 and 2.4, are the Hölder continuous functions. Indeed, take a functions $f \in \mathcal{C}^\gamma$ for some $\gamma > 0$. Using the Taylor expansion of order $\lfloor \gamma \rfloor$, one can associate to f a map \hat{f} with valued in T via

$$\hat{f}: \mathbb{R}^d \rightarrow \bigoplus_{\alpha \in A, \alpha < \gamma} T_\alpha \subset T \quad \text{with} \quad \hat{f}(x) := \sum_{k \in \mathbb{N}, |k| < \gamma} \frac{\partial^k f(x)}{k!} X^k.$$

Equipping a suitable subspace of functions of the form

$$\hat{f}: \mathbb{R}^d \rightarrow \bigoplus_{\alpha \in A, \alpha < \gamma} T_\alpha$$

with the right topology, the map $f \mapsto \hat{f}$ turns out to be an one-to-one correspondence as proven in Lemma 2.12 in [Hai14].

Finally, to enter a more general setting, we need to introduce a local version of the Besov space $\mathcal{B}_{\infty, \infty}^\alpha$. Note that for any non-integer $\alpha > 0$ the Besov space $\mathcal{B}_{\infty, \infty}^\alpha$ coincides with the classical Hölder space \mathcal{C}^α , see for instance [Tri10] or [BCD11].

Definition 2.5 (Definition 3.7 in [Hai14]). Let $\alpha < 0$ and let $r = -[\alpha]$. A distribution $\xi \in \mathcal{D}'$ belongs to the class \mathcal{C}^α if ξ is in the dual of \mathcal{C}_0^r and, for every compact set $\mathcal{K} \subset \mathbb{R}^d$, there exists a constant $C > 0$ such that

$$\langle \xi, \mathcal{S}_x^\delta \eta \rangle \leq C \delta^\alpha$$

holds for all $\eta \in \mathcal{B}^r$, all $\delta \in (0, 1]$, and all $x \in \mathcal{K}$.

2.1 Reconstruction Theorem

From now on we fix an arbitrary regularity structure $\mathcal{T} = (A, T, G)$ with an associated model (Γ, Π) . In this general context the “Hölder continuous” functions relative to the given model are the so-called “modelled distributions”. This class of distributions locally “looks like” the distributions in the model. The precise definition reads as follows.

Definition 2.6 (Definition 3.1 in [Hai14]). Let $\gamma \in \mathbb{R}$. The space of *modelled distributions* \mathcal{D}^γ is given by all functions $f: \mathbb{R}^d \rightarrow T_\gamma^-$ such that for every compact set $\mathcal{K} \subset \mathbb{R}^d$ one has

$$\|f\|_{\gamma, \mathcal{K}} := \sup_{x \in \mathcal{K}} \sup_{\alpha \in A_\gamma} \|f(x)\|_\alpha + \sup_{\substack{x, y \in \mathcal{K} \\ \|x - y\| \leq 1}} \sup_{\alpha \in A_\gamma} \frac{\|f(x) - \Gamma_{x, y} f(y)\|_\alpha}{\|x - y\|^{\gamma - \alpha}} < \infty.$$

Here, we used the notation $T_\gamma^- := \bigoplus_{\alpha \in A_\gamma} T_\alpha$, where one denotes $A_\gamma := \{\alpha \in A : \alpha < \gamma\}$.

The most fundamental result in Martin Hairer’s theory of regularity structures is the reconstruction theorem (Theorem 3.10 in [Hai14]): for every $f \in \mathcal{D}^\gamma$ with $\gamma > 0$ there exists a unique distribution $\mathcal{R}f$ on \mathbb{R}^d such that $\mathcal{R}f$ “looks like” $\Pi_x f(x)$ near x for every $x \in \mathbb{R}^d$. In other words, it is always possible to obtain from an abstract map $f \in \mathcal{D}^\gamma$ a concrete distribution $\mathcal{R}f$, which looks locally in same sense like f .

However, from a probabilist’s point of view it seems much more desirable to work with \mathcal{L}^p -type norms instead of \mathcal{L}^∞ -norms. This has the great advantage to rely on strong and highly developed techniques as stochastic integration. Therefore, we introduce analogously to the classical Sobolev-Slobodeckij spaces (also called fractional Sobolev spaces) for functions $f: \mathbb{R}^d \rightarrow T_\gamma^-$, the norms

$$\|f\|_{\beta, \mathcal{K}, p} := \left(\int_{\mathcal{K}} \|f(x)\|_\beta^p dx \right)^{\frac{1}{p}}$$

and

$$\begin{aligned} \|f\|_{\gamma,p,\mathcal{K}} &:= \left(\sum_{\alpha < \gamma} \|f\|_{\alpha,\overline{\mathcal{K}},p}^p \right)^{\frac{1}{p}} \\ &\quad + \left(\sum_{\alpha < \gamma} \int_{\mathcal{K}} \int_{\mathcal{B}_1(0)} \left(\frac{\|f(x+h) - \Gamma_{x+h,x} f(x)\|_{\alpha}}{\|h\|^{\gamma-\alpha+d/p}} \right)^p dh dx \right)^{\frac{1}{p}}, \end{aligned}$$

for $p \geq 1$, $\gamma \in \mathbb{R}$ and a Borel measurable set $\mathcal{K} \subset \mathbb{R}^d$. Here $\overline{\mathcal{K}}$ stands for the 1-fattening of \mathcal{K} and we shortened the notation by just writing $\sum_{\alpha < \gamma}$ with the meaning $\sum_{\alpha \in A_{\gamma}}$.

Definition 2.7. Let $p \geq 1$ and $\gamma \in \mathbb{R}$. The space of *Sobolev-Slobodeckij type modelled distributions* \mathcal{D}_p^{γ} consists of all functions $f: \mathbb{R}^d \rightarrow T_{\gamma}^{-}$ such that $\|f\|_{\gamma,p,\mathcal{K}} < \infty$ for every compact set $\mathcal{K} \subset \mathbb{R}^d$. Furthermore, for a fixed Borel measurable set $\mathcal{K} \subset \mathbb{R}^d$, we introduce $\mathcal{D}_p^{\gamma}(\mathcal{K})$ as the space of all functions $f: \overline{\mathcal{K}} \rightarrow T_{\gamma}^{-}$ such that $\|f\|_{\gamma,p,\mathcal{K}} < \infty$.

Remark 2.8. We would like to point out that Martin Hairer and Cyril Labbé introduced in [HL15] a different partially L^p -counterpart of the space of modelled distributions \mathcal{D}^{γ} (see Definition 2.9 in [HL15]). Their new spaces $\mathcal{D}^{\gamma,p}$ are in some respect related to Nikolskii spaces (or in other words to Besov spaces $B_{p,\infty}^{\gamma}$), whereas our spaces \mathcal{D}_p^{γ} correspond to an abstract version of Sobolev-Slobodeckij spaces (or the Besov spaces $B_{p,p}^{\gamma}$). Most recently in [HL16] Martin Hairer and Cyril Labbé finally prove general embedding theorems for general Besov type modelled distributions, from where also our first result can be partially deduced. Still our proof of the reconstruction theorem might have some independent interest.

It is easy to check that $\mathcal{D}^{\gamma_1} \subset \mathcal{D}^{\gamma_2}$ and $\mathcal{D}_p^{\gamma_1} \subset \mathcal{D}_p^{\gamma_2}$ for $\gamma_1 \geq \gamma_2$ and $p \in [1, \infty)$. Moreover, one has the following simple embeddings.

Lemma 2.9. Let $\gamma \in \mathbb{R}$, $p \in [1, \infty)$ and $\mathcal{K} \subset \mathbb{R}^d$ be a compact set. The space $\mathcal{D}^{\gamma+d/p}$ embeds continuously into \mathcal{D}_p^{γ} .

Proof. Let us take a function $f \in \mathcal{D}^{\gamma+d/p}$. Since the set $\{\alpha \in A : \alpha < \gamma\}$ has finitely many elements and $\overline{\mathcal{K}} \subset \mathbb{R}^d$ is a compact set, we have

$$\begin{aligned} \|f\|_{\gamma,p,\mathcal{K}} &\lesssim \left(\sum_{\alpha < \gamma} |\overline{\mathcal{K}}| \sup_{x \in \overline{\mathcal{K}}} \|f(x)\|_{\alpha}^p \right)^{\frac{1}{p}} + \left(\sum_{\alpha < \gamma} |\overline{\mathcal{K}}| \left(\sup_{\substack{x,y \in \overline{\mathcal{K}} \\ \|x-y\| \leq 1}} \frac{\|f(x) - \Gamma_{x,y} f(y)\|_{\alpha}}{\|x-y\|^{\gamma-\alpha+d/p}} \right)^p \right)^{\frac{1}{p}} \\ &\lesssim |\overline{\mathcal{K}}|^{\frac{1}{p}} |A_{\gamma}|^{\frac{1}{p}} \|f\|_{\gamma+d/p,\overline{\mathcal{K}}}, \end{aligned}$$

where $|\overline{\mathcal{K}}|$ denotes the volume of $\overline{\mathcal{K}}$ and $|A_{\gamma}|$ the number of elements in A_{γ} . \square

In the next theorem Martin Hairer's celebrated reconstruction theorem is prove for the L^p -type spaces \mathcal{D}_p^{γ} .

Theorem 2.10. Let $\mathcal{T} = (A, T, G)$ be a regularity structure with a model (Π, Γ) on \mathbb{R}^d and let $r > |\alpha_0| + d/p$ with $\alpha_0 := \inf A$ and $p \geq 1$. Then for every $\gamma > d/p$

there exists a linear map $\mathcal{R}: \mathcal{D}_p^\gamma \rightarrow \mathcal{D}'$ such that for every compact set $\mathcal{K} \subset \mathbb{R}^d$ it holds

$$|\langle \mathcal{R}f - \Pi_x f(x), S_x^\delta \eta \rangle| \lesssim \delta^{\gamma-d/p} \|\Pi\|_{\gamma, \mathcal{K}} \|f\|_{\gamma, p, \mathcal{K}} \quad (2)$$

uniformly over all test function $\eta \in \mathcal{B}_0^r$, a $\delta \in (0, \delta_0]$, all $x \in \mathcal{K}$ and for every $f \in \mathcal{D}_p^\gamma \cap C(\mathbb{R}^d, T_\gamma^-)$, where $\delta_0 \in]0, 1]$ depends on f . The operator \mathcal{R} is bounded linear for $\gamma > d/p$, and, of course, all elements $f \in \mathcal{D}_p^\gamma$ are continuous in this case, moreover in all cases $\mathcal{R}f \in \mathcal{C}^{\alpha_0-d/p}$. If $\gamma > d/p$, then the linear map \mathcal{R} is unique.

Before we continue with the proof of Theorem 2.10, some preliminary discussion about wavelet analysis is in order. Let $r > 0$ be a finite real number. We consider a wavelet basis associated to a scaling function $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ with the following four properties:

- (i) The function φ is in \mathcal{C}_0^r .
- (ii) For every polynomial P of degree r , there exists a polynomial \hat{P} of degree r such that $\sum_{y \in \mathbb{Z}^d} \hat{P}(y) \varphi(x-y) = P(x)$ for every $x \in \mathbb{R}^d$.
- (iii) For every $y \in \mathbb{Z}^d$ one has $\int_{\mathbb{R}^d} \varphi(x) \varphi(x-y) dx = \delta_{y,0}$.
- (iv) There exist coefficients $(a_k)_{k \in \mathbb{Z}^d}$ such that

$$2^{-d/2} \varphi(x/2) = \sum_{k \in \mathbb{Z}^d} a_k \varphi(x-k).$$

The existence of such a function φ can be found in Theorem 13.25 in [FH14] and was originally ensured by Daubechies [Dau88]. We set $\varphi_x^n(\cdot) := 2^{nd/2} \varphi(2^n(\cdot-x))$ for $x \in \Lambda^n := 2^{-n}\mathbb{Z}^d$ and thus $(\varphi_x^n)_{x \in \Lambda^n}$ forms an orthonormal family in $L^2(\mathbb{R}^d)$ for every $n \in \mathbb{N}$. The linear span of $(\varphi_x^n)_{x \in \Lambda^n}$ is denote by $V_n \subset \mathcal{C}^r$ and the L^2 -orthogonal complement of V_{n-1} in V_n is denoted by \hat{V}_n . The subspaces \hat{V}_n can be likewise described than the subspaces V_n . Indeed, it is a standard fact coming from wavelet analysis [Mey92]: there exists a finite set Φ of functions such that \hat{V}_{n+1} is the linear span of $(\hat{\varphi}_x^n)_{x \in \Lambda^n; \hat{\varphi} \in \Phi}$, where again $\hat{\varphi}_x^n(\cdot) := 2^{nd/2} \hat{\varphi}(2^n(\cdot-x))$ for $\hat{\varphi} \in \Phi$. Furthermore, the functions in Φ have also compact support and every $n \in \mathbb{N}$, the set

$$\{\varphi_x^n : x \in \Lambda_n\} \cup \{\hat{\varphi}_x^m : \hat{\varphi} \in \Phi, x \in \Lambda_m, m \geq n\}$$

constitute an orthonormal basis of $L^2(\mathbb{R}^d)$. For a more detailed discussion see Section 3.1. in [Hai14].

The proof works similarly to the one of the original reconstruction theorem (cf. Theorem 2.10 in [Hai15] or Theorem 13.26 in [FH14]). However, the relevant estimates are slightly different for the spaces \mathcal{D}_p^γ than for the originally considered spaces \mathcal{D}^γ . Whence we decided to spell out all the details.

Proof of Theorem 2.10. Let $\gamma > d/p$ and $r > |\alpha_0| + d/p$ but finite. Fix a scaling function $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ with the properties (i)-(iv). Choose a function $f \in \mathcal{D}_p^\gamma$, which is by $\gamma > d/p$ continuous. For each $n \in \mathbb{N}$ we define a sequence of operators $\mathcal{R}^n: \mathcal{D}_p^\gamma \rightarrow \mathcal{D}'$ by

$$f \mapsto (\mathcal{R}^n f)(y) := \sum_{x \in \Lambda^n} \langle \Pi_x f(x), \varphi_x^n \rangle \varphi_x^n(y), \quad y \in \mathbb{R}^d.$$

The aim is now to show the convergence of $(\mathcal{R}^n f)_{n \in \mathbb{N}}$ to a distribution $\mathcal{R}f$ such that one has $\langle \mathcal{R}^n f, \psi \rangle \rightarrow \langle \mathcal{R}f, \psi \rangle$ for every test function $\psi \in \mathcal{C}_0^\infty$. Moreover, $\mathcal{R}f$ fulfills the bound (2), which in particular determines $\mathcal{R}f$ uniquely. Under our assumption $\gamma > d/p$ the modelled distribution is continuous, so point evaluations are no problem. Notice the remarkable fact that $f \mapsto \mathcal{R}f$ is continuous with respect to the Sobolev-Slobodeckij norms since in the definition of \mathcal{R} point evaluations appear.

1. *Convergence of \mathcal{R}^n* : First observe the simple identity

$$\begin{aligned} \varphi_x^n(y) &= 2^{nd/2} \varphi(2^n(y-x)) = 2^{nd/2} \sum_{k \in \mathbb{Z}} a_k 2^{d/2} \varphi(2^{n+1}(y-x) - k) \\ &= \sum_{k \in \Lambda^{n+1}} a_{2^n k} \varphi_{x+k}^{n+1}(y), \end{aligned}$$

for $y \in \mathbb{R}^d$, $x \in \Lambda^n$ and $n \in \mathbb{N}$, which holds due to the definition of φ_y^n and property (iv) of φ . For a given $x \in \Lambda^n$, we use this observation together with property (iii) to see that

$$\begin{aligned} \langle \mathcal{R}^{n+1} f - \mathcal{R}^n f, \varphi_x^n \rangle &= \sum_{k \in \Lambda^{n+1}} a_{2^n k} \langle \mathcal{R}^{n+1} f, \varphi_{x+k}^{n+1} \rangle - \langle \Pi_x f(x), \varphi_x^n \rangle \\ &= \sum_{k \in \Lambda^{n+1}} a_{2^n k} \langle \Pi_{x+k} f(x+k), \varphi_{x+k}^{n+1} \rangle - \langle \Pi_x f(x), \varphi_x^n \rangle \\ &= \sum_{k \in \Lambda^{n+1}} a_{2^n k} (\langle \Pi_{x+k} f(x+k), \varphi_{x+k}^{n+1} \rangle - \langle \Pi_x f(x), \varphi_{x+k}^{n+1} \rangle) \\ &= \sum_{k \in \Lambda^{n+1}} a_{2^n k} \langle \Pi_{x+k} (f(x+k) - \Gamma_{x+k,x} f(x)), \varphi_{x+k}^{n+1} \rangle, \end{aligned}$$

where the algebraic relations between Π_x and $\Gamma_{x,y}$ were used for the last equality. Since φ has compact support, there exists a constant $R > 0$ such that $\text{supp } \varphi \subset \mathcal{B}_R(0)$. That means, (iv) also implies that all a_k for $k \in \mathbb{Z}^d \setminus \mathcal{B}_{2R}(0)$ can be choose identically to zero and especially $\max_{k \in \mathbb{Z}^d \cap \mathcal{B}_{2R}(0)} |a_k|$ is finite. In the following the notation $\Lambda_{2R}^n := 2^{-n}(\mathbb{Z}^d \cap \mathcal{B}_{2R}(0))$ is auxiliary. Recall that the definition of a model gives for $\tau \in T_\alpha$ the bound

$$|\langle \Pi_x \tau, \varphi_x^n \rangle| = 2^{-\frac{nd}{2}} |\langle \Pi_x \tau, \mathcal{S}_x^{2^{-n}} \varphi \rangle| \lesssim 2^{-\alpha n - \frac{nd}{2}} \|\tau\|_\alpha \quad (3)$$

uniformly over $n \geq 0$ and $x \in \mathcal{K}$ and thus keeping in mind the above discussion we get

$$\begin{aligned} |\langle \mathcal{R}^{n+1} f - \mathcal{R}^n f, \varphi_x^n \rangle| &\leq \left| \sum_{k \in \Lambda^{n+1}} a_{2^n k} \langle \Pi_{x+k} (f(x+k) - \Gamma_{x+k,x} f(x)), \varphi_{x+k}^{n+1} \rangle \right| \\ &\leq \sum_{k \in \Lambda_{2R}^{n+1}} |a_{2^n k}| |\langle \Pi_{x+k} f(x+k) - \Gamma_{x+k,x} f(x), \varphi_{x+k}^{n+1} \rangle| \\ &\lesssim \sum_{k \in \Lambda_{2R}^{n+1}} \sum_{\alpha < \gamma} |a_{2^n k}| 2^{-\alpha n - \frac{nd}{2}} \|f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha. \end{aligned}$$

Applying Hlder's inequality with $1/p + 1/q = 1$ leads to

$$\begin{aligned}
& |\langle \mathcal{R}^{n+1}f - \mathcal{R}^n f, \varphi_x^n \rangle| \\
& \lesssim \sum_{k \in \Lambda_{2R}^{n+1}} \sum_{\alpha < \gamma} |a_{2^n k}| 2^{-\alpha n - \frac{nd}{2}} \frac{\|f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha}{\|k\|^{\gamma-\alpha}} 2^{-(\gamma-\alpha)n} \\
& \leq 2^{-\gamma n - \frac{nd}{2}} \left(\sum_{k \in \Lambda_{2R}^{n+1}} \sum_{\alpha < \gamma} |a_{2^n k}|^q \right)^{1/q} \\
& \quad \times \left(\sum_{k \in \Lambda_{2R}^{n+1}} \sum_{\alpha < \gamma} \left(\frac{\|f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha}{\|k\|^{\gamma-\alpha}} \right)^p \right)^{\frac{1}{p}} \\
& \leq 2^{-\gamma n - \frac{nd}{2}} \left(\sum_{k \in \Lambda_{2R}^{n+1}} \sum_{\alpha < \gamma} \left(\frac{\|\Pi_{x+k} f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha}{\|k\|^{\gamma-\alpha}} \right)^p \right)^{\frac{1}{p}},
\end{aligned}$$

where we used in the last line $\max_{k \in \mathbb{Z}^d \cap \mathcal{B}_{2R}(0)} |a_k| < \infty$ and that only finitely many summands are non-zero. For an arbitrary test function $\psi \in \mathcal{C}^r$ with compact support in \mathcal{K} , we want to estimate $\langle \mathcal{R}^{n+1}f - \mathcal{R}^n f, \psi \rangle$. Since $\mathcal{R}^{n+1}f - \mathcal{R}^n f \in V_{n+1}$, we can decompose the difference into a part $\delta \mathcal{R}^n \in V_n$ and a part $\hat{\delta} \mathcal{R} \in \hat{V}_{n+1}$, which we shall estimate separately. In the following we take $n \in \mathbb{N}$ big enough to ensure that $\text{supp } \varphi_0^n \subset \mathcal{B}_1(0)$. Concerning the part in V_n , we have

$$\begin{aligned}
& |\langle \delta \mathcal{R}^n f, \psi \rangle| = \left| \sum_{x \in \Lambda^{n+1}} \langle \delta \mathcal{R}^n f, \varphi_x^n \rangle \langle \varphi_x^n, \psi \rangle \right| \\
& \leq \left(\sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} |\langle \delta \mathcal{R}^n f, \varphi_x^n \rangle|^p \right)^{\frac{1}{p}} \left(\sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} |\langle \varphi_x^n, \psi \rangle|^q \right)^{\frac{1}{q}} \\
& \lesssim 2^{-\gamma n - \frac{nd}{2}} 2^{\frac{nd}{p}} \times 2^{-\frac{nd}{2}} 2^{\frac{nd}{q}} \\
& \quad \times \left(\sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} \sum_{k \in \Lambda_{2R}^{n+1}} \sum_{\alpha < \gamma} \left(\frac{\|f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha}{\|k\|^{\gamma-\alpha}} \right)^p \right)^{\frac{1}{p}} \\
& \lesssim 2^{-(\gamma-d/p)n} \left(\sum_{\alpha < \gamma} \sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} \sum_{k \in \Lambda_{2R}^{n+1}} \left(\frac{\|f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha}{\|k\|^{\gamma-\alpha+d/p}} \right)^p 2^{-2nd} \right)^{\frac{1}{p}}.
\end{aligned} \tag{4}$$

Regarding the second term $\hat{\delta} \mathcal{R}$, we recall that there exists a finite set Φ of functions such that \hat{V}_{n+1} is the linear span of $(\hat{\varphi}_x^n)_{x \in \Lambda^n; \hat{\varphi} \in \Phi}$ and thanks to (ii) every $\hat{\varphi} \in \Phi$ satisfies

$$\int_{\mathbb{R}^d} \hat{\varphi}(x) P(x) dx = 0 \tag{5}$$

for any polynomial P of degree less or equal to r . In particular, this implies that

$$|\langle \hat{\varphi}_x^n, \psi \rangle| \lesssim 2^{-\frac{nd}{2} - nr}.$$

Further, we may choose $R > 0$ big enough such that also $\text{supp } \hat{\varphi} \subset \mathcal{B}_R(0)$ for

every $\hat{\varphi} \in \Phi$. With these facts at hand, it follows

$$|\langle \hat{\delta} \mathcal{R}^n f, \psi \rangle| = \left| \sum_{\substack{x \in \Lambda^n \cap \bar{\mathcal{K}} \\ \hat{\varphi} \in \Phi}} \langle \mathcal{R}^{n+1} f, \hat{\varphi}_x^n \rangle \langle \hat{\varphi}_x^n, \psi \rangle \right| \lesssim 2^{-\frac{nd}{2} - nr} \sum_{\substack{x \in \Lambda^n \cap \bar{\mathcal{K}} \\ \hat{\varphi} \in \Phi}} |\langle \mathcal{R}^{n+1} f, \hat{\varphi}_x^n \rangle|.$$

For a given $y \in \Lambda^n \cap \bar{\mathcal{K}}$, each summand $|\langle \mathcal{R}^{n+1} f, \hat{\varphi}_y^n \rangle|$ can be further estimated

$$\begin{aligned} |\langle \mathcal{R}^{n+1} f, \hat{\varphi}_y^n \rangle| &\leq \sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} |\langle \Pi_x f(x), \varphi_x^{n+1} \rangle| |\langle \varphi_x^{n+1}, \hat{\varphi}_y^n \rangle| \\ &\lesssim \sum_{\alpha < \gamma} \sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} |\langle \varphi_x^{n+1}, \hat{\varphi}_y^n \rangle| 2^{-\alpha n - \frac{nd}{2}} \|f(x)\|_\alpha \\ &\lesssim 2^{-\alpha n - \frac{nd}{2}} \left(\sum_{\alpha < \gamma} \sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} |\langle \varphi_x^{n+1}, \hat{\varphi}_y^n \rangle|^q \right)^{\frac{1}{q}} \left(\sum_{\alpha < \gamma} \sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} \|f(x)\|_\alpha^p \right)^{\frac{1}{p}} \\ &\lesssim 2^{-\alpha_0 n + \frac{nd}{2} + \frac{nd}{p}} \left(\sum_{\alpha < \gamma} \sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} \|f(x)\|_\alpha^p 2^{-nd} \right)^{\frac{1}{p}}, \end{aligned}$$

where the definition of a model was used in the second line and Hölder's inequality with $1/q + 1/p = 1$ in the third. Therefore, one get

$$|\langle \hat{\delta} \mathcal{R}^n f, \psi \rangle| \lesssim 2^{(-\alpha_0 - r + \frac{d}{p})n} \left(\sum_{\alpha < \gamma} \sum_{x \in \Lambda^{n+1} \cap \bar{\mathcal{K}}} \|f\|_\alpha^p 2^{-nd} \right)^{\frac{1}{p}}. \quad (6)$$

Therefore, thanks to the estimates (4) and (6), one has indeed $\mathcal{R}^n f \rightarrow \mathcal{R}f$ as $n \rightarrow \infty$ for some distribution $\mathcal{R}f$.

2. *Bound (2):* For a given distribution $\eta \in \mathcal{C}^\alpha$ for some $\alpha > -r$, we set

$$\mathcal{P}_n \eta := \sum_{x \in \Lambda^n} \eta(\varphi_x^n) \varphi_x^n \quad \text{and} \quad \hat{\mathcal{P}}_n \eta := \sum_{\hat{\varphi} \in \Phi} \sum_{x \in \Lambda^n} \eta(\hat{\varphi}_x^n) \hat{\varphi}_x^n.$$

In particular, for every $f \in \mathcal{D}_p^\gamma$ one can rewrite $\Pi_x f(x)$ as

$$\Pi_x f(x) = \mathcal{P}_n \Pi_x f(x) + \sum_{m \geq n} \hat{\mathcal{P}}_m \Pi_x f(x). \quad (7)$$

Let us fix a $\lambda \in (0, \delta_0]$, where $\delta_0 \in]0, 1]$ is chosen such that approximations on the grid Λ^n of our norms $\|f\|_{\gamma, p, \mathcal{K}}$ are close up to $\|f\|_{\gamma, p, \mathcal{K}}$ for $2^{-n} \leq \delta_0$. We choose $n \in \mathbb{N}$ such that $2^{-n} \sim \lambda$. Keeping (7) in mind, one has identity

$$\begin{aligned} \mathcal{R}f - \Pi_x f(x) &= \mathcal{R}^n f - \mathcal{P}_n \Pi_x f(x) + \sum_{m \geq n} (\mathcal{R}^{m+1} f - \mathcal{R}^m f - \hat{\mathcal{P}}_m \Pi_x f(x)) \\ &= \mathcal{R}^n f - \mathcal{P}_n \Pi_x f(x) + \sum_{m \geq n} (\hat{\delta} \mathcal{R}^m f - \hat{\mathcal{P}}_m \Pi_x f(x)) + \sum_{m \geq n} \delta \mathcal{R}^m f. \quad (8) \end{aligned}$$

Now we test these terms against ψ_x^λ and deal with the three in (8) appearing terms separately. Due to the definition of \mathcal{R}^n and \mathcal{P}_n , the first term in (8) can be rewritten as

$$\langle \mathcal{R}^n f - \mathcal{P}_n \Pi_x f(x), \psi_x^\lambda \rangle = \sum_{y \in \Lambda^n} \langle \Pi_y f(y) - \Pi_x f(x), \varphi_y^n \rangle \langle \varphi_y^n, \psi_x^\lambda \rangle. \quad (9)$$

Notice that we have the bounds $|\langle \varphi_y^n, \psi_x^n \rangle| \lesssim \lambda^{-d} 2^{-dn/2} \sim 2^{dn/2}$ and $|y-x| \lesssim \lambda$ for all non-vanishing terms in the above sum. By the definition of a model and especially recalling (3), we obtain

$$\begin{aligned} |\langle \Pi_y f(y) - \Pi_x f(x), \varphi_y^n \rangle| &= |\langle \Pi_y(f(y) - \Gamma_{y,x} f(x)), \varphi_y^n \rangle| \\ &\lesssim \sum_{\alpha < \gamma} \lambda^{\gamma-\alpha} 2^{-\frac{dn}{2}-\alpha n} \sim 2^{-\frac{dn}{2}-\gamma n}. \end{aligned}$$

Since only finitely many (independently of n) terms contribute to the sum in (9), we deduce

$$|\langle \mathcal{R}^n f - \mathcal{P}_n \Pi_x f(x), \psi_x^\lambda \rangle| \lesssim 2^{-\gamma n} \sim \lambda^\gamma.$$

For the second term in (8) we start by considering a fixed $m \geq n$ and reformulate this term as

$$\begin{aligned} &\langle \hat{\delta} \mathcal{R}^m f - \hat{\mathcal{P}}_m \Pi_x f(x), \psi_x^\lambda \rangle \\ &= \sum_{\hat{\varphi} \in \Phi} \sum_{y \in \Lambda^{m+1} z \in \Lambda^m} \langle \Pi_y f(y) - \Pi_x f(x), \varphi_y^{m+1} \rangle \langle \varphi_y^{m+1}, \hat{\varphi}_z^m \rangle \langle \hat{\varphi}_z^m, \psi_x^\lambda \rangle. \end{aligned}$$

Here we use that property (5) of $\hat{\varphi}$ provides the bound

$$|\langle \hat{\varphi}_y^m, \psi_x^\lambda \rangle| \lesssim \lambda^{-d-r} 2^{-rm-\frac{md}{2}},$$

and the L^2 -scaling implies $|\langle \varphi_y^m, \hat{\varphi}_z^m \rangle| \lesssim 1$. Moreover, for every $z \in \Lambda^m$, there are only finitely many elements $y \in \Lambda^{m+1}$ contributing to the sum, and these elements all fulfill $|y-z| \lesssim 2^{-m}$. As in the original proof, we end up for the second sum in (8) with the bound given by

$$\sum_{m \geq n} \lambda^d 2^{md} \sum_{\alpha < \gamma} \lambda^{\gamma-\alpha-d-r} 2^{-dm-\alpha m-rm} \sim \sum_{\alpha < \gamma} \lambda^{\gamma-\alpha-r} \sum_{m \geq n} 2^{-\alpha m-rm} \sim \lambda^\gamma.$$

For the third term in (8), using $|\langle \varphi_y^m, \psi_x^\lambda \rangle| \lesssim \lambda^{-d} 2^{-dm/2}$ and similar argument as in (4), we have

$$\begin{aligned} &|\langle \delta \mathcal{R}^m f, \psi_x^\lambda \rangle| \\ &\leq \left(\sum_{y \in \Lambda^{m+1}} |\langle \delta \mathcal{R}^m, \varphi_y^m \rangle|^p \right)^{1/p} \left(\sum_{y \in \Lambda^{m+1}} |\langle \varphi_y^m, \psi_x^\lambda \rangle|^q \right)^{\frac{1}{q}} \\ &\lesssim \lambda^{\frac{d}{q}} 2^{\frac{dm}{q}} \lambda^{-d} 2^{-\frac{dm}{2}} 2^{-\gamma m - \frac{dm}{2}} \\ &\quad \times \left(\sum_{x \in \Lambda^{m+1} \cap \mathcal{K}} \sum_{k \in \Lambda_1^{m+1}} \sum_{\alpha < \gamma} \left(\frac{\|f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha}{\|k\|^{\gamma-\alpha+d/p}} \right)^p 2^{-dm} \right)^{\frac{1}{p}} \\ &\lesssim 2^{-\gamma m} \lambda^{-\frac{d}{p}} \\ &\quad \times \left(\sum_{x \in \Lambda^{m+1} \cap \mathcal{K}} \sum_{k \in \Lambda_1^{m+1}} \sum_{\alpha < \gamma} \left(\frac{\|f(x+k) - \Gamma_{x+k,x} f(x)\|_\alpha}{\|k\|^{\gamma-\alpha+d/p}} \right)^p 2^{-2dm} \right)^{\frac{1}{p}}, \end{aligned}$$

for every $m \geq n$ and $1/p + 1/q = 1$. Combing the last two bounds, we get

$$\sum_{m \geq n} |\langle \delta \mathcal{R}^m f, \psi_x^\lambda \rangle| \lesssim \|f\|_{\gamma,p,\mathcal{K}} \sum_{m \geq n} 2^{-\gamma m} \lambda^{-\frac{d}{p}} \sim 2^{-\gamma n} \lambda^{-\frac{d}{p}} \sim \lambda^{\gamma-\frac{d}{p}}.$$

Putting all bounds for the three terms together, we end up with the claimed inequality (2).

3. *Uniqueness of \mathcal{R}* : Suppose there exist two distributions $\mathcal{R}f$ and $\tilde{\mathcal{R}}f$ for some $f \in \mathcal{D}_p^\gamma$ such that both fulfill the bound (2). Let $\psi: \mathbb{R}^d \rightarrow \mathbb{R}$ be an arbitrary smooth function with compact support in \mathcal{K} and take an other smooth function $\rho: \mathcal{B}_1(0) \rightarrow [0, \infty)$ with $\int_{\mathbb{R}^d} \rho(x) dx = 1$. It is easy to check that

$$\psi_\delta(y) := \int_{\mathbb{R}^d} \psi(x) (\mathcal{S}_x^\delta \rho)(y) dx \rightarrow \psi(y) \quad \text{as } \delta \rightarrow 0,$$

where the convergence take place in the \mathcal{C}^∞ topology. Therefore, by applying bound (2) twice, one sees that

$$\begin{aligned} |\langle \mathcal{R}f - \tilde{\mathcal{R}}f, \psi \rangle| &= \lim_{\delta \rightarrow 0} \left| \int_{\mathbb{R}^d} \psi(x) \langle \mathcal{R}f - \tilde{\mathcal{R}}f, \mathcal{S}_x^\delta \rho \rangle dx \right| \\ &\leq \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^d} |\psi(x)| (|\langle \mathcal{R}f - \Pi_x f(x), \mathcal{S}_x^\delta \rho \rangle| + |\langle \tilde{\mathcal{R}}f - \Pi_x f(x), \mathcal{S}_x^\delta \rho \rangle|) dx \\ &\lesssim \|\Pi\|_{\gamma, \mathcal{K}} \|f\|_{\gamma, p, \mathcal{K}} \left(\int_{\mathbb{R}^d} |\psi(x)| dx \right) \lim_{\delta \rightarrow 0} \delta^{\gamma - \frac{d}{p}} = 0. \end{aligned}$$

This implies that $|\langle \mathcal{R}f - \tilde{\mathcal{R}}f, \psi \rangle| = 0$ for all smooth compactly support test functions and such $\mathcal{R}f - \tilde{\mathcal{R}}f \equiv 0$.

4. *$\mathcal{R}f$ is of class $\mathcal{C}^{\alpha_0 - d/p}$* : Since $r > |\alpha_0| + d/p$ and the convergence as shown in 1. took place in the dual of \mathcal{C}_0^r . It remains to show that $\mathcal{R}f$ decreases of order $\alpha_0 - d/p$. Taking $\eta \in \mathcal{B}^r$ we apply the definition of a model and again bound (2) to obtain

$$\begin{aligned} |\langle \mathcal{R}f, \mathcal{S}_x^\delta \eta \rangle| &\leq |\langle \mathcal{R}f - \Pi_x f(x), \mathcal{S}_x^\delta \eta \rangle| + |\langle \Pi_x f(x), \mathcal{S}_x^\delta \eta \rangle| \\ &\lesssim \|\Pi\|_{\gamma, \mathcal{K}} \|f\|_{\gamma, p, \mathcal{K}} \delta^{\gamma - \frac{d}{p}} + \sum_{\alpha < \gamma} \delta^\alpha \lesssim \delta^{\alpha_0 - \frac{d}{p}}. \end{aligned}$$

□

3 Stochastic Integration on \mathcal{D}_p^γ

This section is devoted to prove that the space of Sobolev-Slobodeckij type modelled distributions \mathcal{D}_p^γ is locally an UMD Banach space and of martingale type 2, which then opens the door to apply highly developed stochastic integration theory on the space \mathcal{D}_p^γ . Having such a property locally just means that the Banach space \mathcal{D}_p^γ appropriately factorized by functions f with vanishing norm $\|f\|_{\gamma, p, \mathcal{K}}$ satisfies the property.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $I \subset \mathbb{R}$, $(\mathcal{F}_t)_{t \in I}$ be an increasing family of sub- σ -algebra of \mathcal{F} and X be a Banach space with norm $\|\cdot\|_X$. A process $(M_t)_{t \in I}$ is a X -valued martingale if and only if $M_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}; X)$ for all $t \in I$ and

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s \quad \mathbb{P}\text{-a.s.}, \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

A sequence $(\xi_i)_{i \in \mathbb{N}}$ is called *martingale difference* if $(\sum_{i=0}^n \xi_i)_{n \in \mathbb{N}}$ is a X -valued martingale. To rely on stochastic integration theory on Banach spaces, one needs

to require some additional properties on the Banach space X . The definitions are taken from [Brz95], see Definition 2.1 and Definition B.2 therein.

Definition 3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

1. A Banach space $(X, \|\cdot\|_X)$ is of *martingale type p* for $p \in [1, \infty)$ if any X -valued martingale $(M_n)_{n \in \mathbb{N}}$ satisfies

$$\sup_n \mathbb{E}[\|M_n\|_X^p] \leq C_p(X) \sum_{n \in \mathbb{N}} \mathbb{E}[\|M_n - M_{n-1}\|_X^p]$$

for some constant $C_p(X) > 0$ independent of the martingale $(M_n)_{n \in \mathbb{N}}$ and $M_{-1} := 0$.

2. A Banach space $(X, \|\cdot\|_X)$ is of *type p* for $p \in [1, 2]$ if any finite sequence $\epsilon_1, \dots, \epsilon_n: \Omega \rightarrow \{-1, 1\}$ of symmetric and i.i.d. random variables and for any finite sequence x_1, \dots, x_n of elements of X the inequality

$$\mathbb{E} \left[\left\| \sum_{i=1}^n \epsilon_i x_i \right\|_X^p \right] \leq K_p(X) \sum_{i=1}^n \|x_i\|_X^p$$

holds for some constant $K_p(X) > 0$.

3. A Banach space $(X, \|\cdot\|_X)$ is called an *UMD space* or is said to have the *unconditional martingale property* if for any $p \in (1, \infty)$, for any martingale difference $(\xi_j)_{j \in \mathbb{N}}$ and for any sequence $(\epsilon_i)_{i \in \mathbb{N}} \subset \{-1, 1\}$ the inequality

$$\mathbb{E} \left[\left\| \sum_{i=1}^n \epsilon_i \xi_j \right\|_X^p \right] \leq \tilde{K}_p(X) \mathbb{E} \left[\left\| \sum_{i=1}^n \xi_i \right\|_X^p \right]$$

holds for all $n \in \mathbb{N}$, where $\tilde{K}_p(X) > 0$ is some constant.

Let us remark that Hilbert spaces and finite dimensional Banach spaces are always UMD spaces. For a more comprehensive introduction and treatment of stochastic integration in Banach spaces we refer for instance to [Dal15, MR15].

Coming back to a regularity structure $\mathcal{T} = (T, G, A)$ with an associated model (Π, Γ) and let us assume now additionally that each T_α is an UMD space for $\alpha \in A$. Under this assumption the space $T_\gamma^- = \bigoplus_{\alpha < \gamma} T_\alpha$ is again an UMD space (Theorem 4.5.2 in [Ama95]) since A is locally finite and T is a finite product of UMD spaces.

Theorem 3.2. *Suppose that the Banach space T_γ^- is an UMD space and $\mathcal{K} \subset \mathbb{R}^d$ is a Borel measurable set. Then, the spaces \mathcal{D}_p^γ and $\mathcal{D}_p^\gamma(\mathcal{K})$ are UMD spaces, too, for $1 < p < \infty$. If the Banach space T_γ^- is additionally of type 2, then \mathcal{D}_p^γ and $\mathcal{D}_p^\gamma(\mathcal{K})$ are of martingale type 2 for every $p \geq 2$.*

Proof. Let $\mathcal{K} \subset \mathbb{R}^d$ be a Borel measurable set. We define the measure space $(\overline{\mathcal{K}} \times \mathcal{B}_1(0), \mathcal{B}(\overline{\mathcal{K}} \times \mathcal{B}_1(0)), d\lambda_{\mathcal{K}} \otimes d\lambda_{\mathcal{B}_1(0)})$, where $\mathcal{B}(\overline{\mathcal{K}} \times \mathcal{B}_1(0))$ denotes the Borel σ -algebra on $\overline{\mathcal{K}} \times \mathcal{B}_1(0)$ and $\lambda_{\overline{\mathcal{K}}}$ resp. $\lambda_{\mathcal{B}_1(0)}$ are the Lebesgue measures on $\overline{\mathcal{K}}$ resp. $\mathcal{B}_1(0)$. Since by assumption T_γ^- is UMD, $L^p(\overline{\mathcal{K}} \times \mathcal{B}_1(0), T_\gamma^-)$ is also UMD,

see Theorem 4.5.2 in [Ama95], for $1 < p < \infty$. We have to understand that \mathcal{D}_p^γ is a closed subspace of the product space

$$L^p(\overline{\mathcal{K}} \times \mathcal{B}_1(0), T_\gamma^-)^{|A_\gamma|} \times L^p(\mathcal{K} \times \mathcal{B}_1(0), T_\gamma^-)^{|A_\gamma|},$$

which is also UMD, where $|A_\gamma|$ denotes the number of elements in A_γ . For this purpose let us now define for every $\alpha \in A_\gamma$ the bounded linear operators

$$\Phi_1^\alpha : \mathcal{D}_p^\gamma \rightarrow L^p(\overline{\mathcal{K}} \times \mathcal{B}_1(0), T_\gamma^-) \quad \text{via} \quad \Phi_1^\alpha(f)(x, h) := f(x), \quad (x, h) \in \overline{\mathcal{K}} \times \mathcal{B}_1(0),$$

and

$$\begin{aligned} \Phi_2^\alpha : \mathcal{D}_p^\gamma &\rightarrow L^p(\mathcal{K} \times \mathcal{B}_1(0), T_\gamma^-) \\ \text{via } \Phi_2^\alpha(f)(x, h) &:= \frac{f(x+h) - \Gamma_{x+h,x}f(x)}{\|h\|^{\gamma+\alpha+d/p}}, \quad (x, h) \in \mathcal{K} \times \mathcal{B}_1(0). \end{aligned}$$

It is sufficient to show that

$$\begin{aligned} \|f\|_{\gamma,p,\mathcal{K}}^2 &\leq C_1 \sum_{j \in A_\gamma} (\|\Phi_1^j(f)\|_{L^p(\overline{\mathcal{K}} \times \mathcal{B}_1(0), T_\gamma^-)}^2 + \|\Phi_1^j(f)\|_{L^p(\mathcal{K} \times \mathcal{B}_1(0), T_\gamma^-)}^2) \\ &\leq C_1 C_2 \|f\|_{\gamma,p,\mathcal{K}}^2, \end{aligned} \tag{10}$$

for $f \in \mathcal{D}_p^\gamma$ and for some constant $C_1, C_2 > 0$, in order to prove the UMD property of \mathcal{D}_p^γ . Indeed, in our situation this can be seen as

$$\begin{aligned} \|f\|_{\gamma,p,\mathcal{K}}^2 &\leq 2 \left(\sum_{\alpha < \gamma} \|f\|_{\alpha,\overline{\mathcal{K}},p}^p \right)^{\frac{2}{p}} \\ &\quad + 2 \left(\sum_{\alpha < \gamma} \int_{\mathcal{K}} \int_{\mathcal{B}^1(0)} \left(\frac{\|f(x+h) - \Gamma_{x+h,x}f(x)\|_\alpha}{\|h\|^{\gamma-\alpha+d/p}} \right)^p dh dx \right)^{\frac{2}{p}} \\ &\leq 2 \sum_{\alpha < \gamma} \|f\|_{\alpha,\overline{\mathcal{K}},p}^2 \\ &\quad + 2 \sum_{\alpha < \gamma} \left(\int_{\mathcal{K}} \int_{\mathcal{B}^1(0)} \left(\frac{\|f(x+h) - \Gamma_{x+h,x}f(x)\|_\alpha}{\|h\|^{\gamma-\alpha+d/p}} \right)^p dh dx \right)^{\frac{2}{p}} \\ &\leq \sum_{j \in \times A_\gamma} (\|\Phi_1^j(f)\|_{L^p(\overline{\mathcal{K}} \times \mathcal{B}_1(0), T_\gamma^-)}^2 + \|\Phi_1^j(f)\|_{L^p(\mathcal{K} \times \mathcal{B}_1(0), T_\gamma^-)}^2) \end{aligned}$$

and

$$\sum_{j \in \times A_\gamma} (\|\Phi_1^j(f)\|_{L^p(\overline{\mathcal{K}} \times \mathcal{B}_1(0), T_\gamma^-)}^2 + \|\Phi_1^j(f)\|_{L^p(\mathcal{K} \times \mathcal{B}_1(0), T_\gamma^-)}^2) \leq 2|A_\gamma|^2 \|f\|_{\gamma,p,\mathcal{K}}^2.$$

Of course we can embed with these maps \mathcal{D}_p^γ into the above product space as a closed subspace. By Theorem 4.5.2 in [Ama95] the space \mathcal{D}_p^γ is therefore UMD, too. The previous construction is similar to Lemma A.5 in [Brz95]. Since every UMD space of type 2 is a Banach space of martingale type 2 as shown in Proposition B.4 in [Brz95], one concludes that $L^p(\overline{\mathcal{K}} \times \mathcal{B}_1(0), T_\gamma^-)$ is a space of martingale type 2 for every $p \in [2, \infty)$, and the same argument as before applies. \square

We can now formulate and prove our main theorem. Like in the Fubini theorem the order of reconstruction and stochastic integration can be interchanged:

Theorem 3.3. *Let $\gamma > d/p$ and $\mathcal{T} = (A, T, G)$ be a regularity structure together with a model (Π, Γ) . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space and W be a Brownian motion. Let H be a bounded modelled distribution valued predictable process, then the order of “integration” can be interchanged*

$$\left\langle \mathcal{R}((H \bullet W)), \psi \right\rangle = \left(\langle \mathcal{R}(H), \psi \rangle \bullet W \right)$$

for every test function $\psi \in \mathcal{B}_0^r$. Here $(H \bullet W)$ denotes the stochastic integral of H with respect to W .

Proof. Since the reconstruction operator \mathcal{R} is continuous on \mathcal{D}_p^γ with respect to the local norms, we can interchange the order of stochastic integration and application of \mathcal{R} . \square

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